Probability Distributions

Dr Azmi Mohd Tamil

Definitions

- Random Variable
  - Variable whose values are determined by chance
- Probability Distribution
  - The values a random variable can assume and the corresponding probabilities of each.
- Expected Value
  - The theoretical mean of the variable.
- Binomial Experiment
  - An experiment with a fixed number of independent trials. Each trial can only have two outcomes, or outcomes which can be reduced to two outcomes. The probability of each outcome must remain constant from trial to trial.

Definitions

- Binomial Distribution
  - The outcomes of a binomial experiment with their corresponding probabilities.
- Multinomial Distribution
  - A probability distribution resulting from an experiment with a fixed number of independent trials. Each trial has two or more mutually exclusive outcomes. The probability of each outcome must remain constant from trial to trial.
- Poisson Distribution
  - A probability distribution used when a density of items is distributed over a period of time. The sample size needs to be large and the probability of success to be small.
- Hypergeometric Distribution
  - A probability distribution of a variable with two outcomes when sampling is done without replacement.

Probability Functions

- A probability function is a function which assigns probabilities to the values of a random variable.
- All the probabilities must be between 0 and 1 inclusive
- The sum of the probabilities of the outcomes must be 1.

Probability Distributions

- A listing of all the values the random variable can assume with their corresponding probabilities make a probability distribution.

probability distribution that results from the rolling of a single fair die.

x	1	2	3	4	5	6	**sum**
p(x)	1/6	1/6	1/6	1/6	1/6	1/6	**6/6=1**

Binomial Distributions

Dr Azmi Mohd Tamil

A binomial experiment satisfies these four conditions;

- A fixed number of trials
- Each trial is independent of the others
- There are only two outcomes
- The probability of each outcome remains constant from trial to trial.

A binomial experiment is

- An experiment with a fixed number of independent trials, each of which can only have two possible outcomes.
- The fact that each trial is independent actually means that the probabilities remain constant.

Examples

- Tossing a coin 20 times to see how many tails occur.
- Asking 200 people if they watch TV3 news.
- Rolling a die to see if a 5 appears.

NOT Binomial Experiments!

- Rolling a die until a 6 appears (not a fixed number of trials)
- Asking 20 people how old they are (not two outcomes)
- Drawing 5 cards from a deck for a poker hand (done without replacement, so not independent)

Binomial Probability Function

Example

What is the probability of rolling exactly two sixes in 6 rolls of a die?

There are five things you need to do to work a binomial story problem.

1. Define Success first. Success must be for a single trial. Success = "Rolling a 6 on a single die"
2. Define the probability of success (p): $p = 1/6$
3. Find the probability of failure: $q = 5/6$
4. Define the number of trials: $n = 6$
5. Define the number of successes out of those trials: $x = 2$

- Anytime a six appears, it is a success (denoted S) and anytime something else appears, it is a failure (denoted F). The ways you can get exactly 2 successes in 6 trials are given below. The probability of each is written to the right of the way it could occur. Because the trials are independent, the probability of the event (all six dice) is the product of each probability of each outcome (die)

1 FFFFSS $5/6 * 5/6 * 5/6 * 5/6 * 1/6 * 1/6 = (1/6)^2 * (5/6)^4$

2 FFFSFS $5/6 * 5/6 * 5/6 * 1/6 * 5/6 * 1/6 = (1/6)^2 * (5/6)^4$

3 FFFSSF $5/6 * 5/6 * 5/6 * 1/6 * 1/6 * 5/6 = (1/6)^2 * (5/6)^4$

4 FFSFFS $5/6 * 5/6 * 1/6 * 5/6 * 5/6 * 1/6 = (1/6)^2 * (5/6)^4$

5 FFSFSF $5/6 * 5/6 * 1/6 * 5/6 * 1/6 * 5/6 = (1/6)^2 * (5/6)^4$

6 FFSSFF $5/6 * 5/6 * 1/6 * 1/6 * 5/6 * 5/6 = (1/6)^2 * (5/6)^4$

7 FSFFFS $5/6 * 1/6 * 5/6 * 5/6 * 5/6 * 1/6 = (1/6)^2 * (5/6)^4$

8 FSFFSF $5/6 * 1/6 * 5/6 * 5/6 * 1/6 * 5/6 = (1/6)^2 * (5/6)^4$

9 FSFSFF $5/6 * 1/6 * 5/6 * 1/6 * 5/6 * 5/6 = (1/6)^2 * (5/6)^4$

10 FSSFFF $5/6 * 1/6 * 1/6 * 5/6 * 5/6 * 5/6 = (1/6)^2 * (5/6)^4$

11 SFFFFS $1/6 * 5/6 * 5/6 * 5/6 * 5/6 * 1/6 = (1/6)^2 * (5/6)^4$

12 SFFFSF $1/6 * 5/6 * 5/6 * 5/6 * 1/6 * 5/6 = (1/6)^2 * (5/6)^4$

13 SFFSFF $1/6 * 5/6 * 5/6 * 1/6 * 5/6 * 5/6 = (1/6)^2 * (5/6)^4$

14 SFSFFF $1/6 * 5/6 * 1/6 * 5/6 * 5/6 * 5/6 = (1/6)^2 * (5/6)^4$

15 SSFFFF $1/6 * 1/6 * 5/6 * 5/6 * 5/6 * 5/6 = (1/6)^2 * (5/6)^4$

- Notice that each of the 15 probabilities are exactly the same: $(1/6)^2 * (5/6)^4$.
- Also, note that the 1/6 is the probability of success and you needed 2 successes. The 5/6 is the probability of failure, and if 2 of the 6 trials were success, then 4 of the 6 must be failures. Note that 2 is the value of x and 4 is the value of n-x.
- Further note that there are fifteen ways this can occur. This is the number of ways 2 successes can be occur in 6 trials without repetition and order not being important, or a combination of 6 things, 2 at a time.

- **The probability of getting exactly x success in n trials, with the probability of success on a single trial being p is:**

$$P(X=x) = {}^{n}C_{x} * p^{x} * q^{(n-x)}$$

Example:

- A coin is tossed 10 times. What is the probability that exactly 6 heads will occur.
- Success = "A head is flipped on a single coin"
- p = 0.5
- q = 0.5
- n = 10
- x = 6

$P(x=6) = {}^{10}C_6 * 0.5^6 * 0.5^4$

$= 210 * 0.015625 * 0.0625$

$= 0.205078125$

Mean, Variance, and Standard Deviation

- The mean, variance, and standard deviation of a binomial distribution are extremely easy to find.

$$\mu = np$$

$$\sigma^2 = npq$$

$$\sigma = \sqrt{npq}$$

Example

Find the mean, variance, and standard deviation for the number of sixes that appear when rolling 30 dice.

- Success = "a six is rolled on a single die".
  p = 1/6,
  q = 5/6.
- The mean is 30 * (1/6) = 5.
  The variance is 30 * (1/6) * (5/6) = 25/6.
  The standard deviation is the square root of the variance = 2.041241452 (approx)

Other Discrete Distributions

Other Discrete Distributions

- **Multinomial Probabilities**
- Poisson Probabilities
- Hypergeometric Probabilities

Multinomial Probabilities

- A multinomial experiment is an extended binomial probability. The difference is that in a multinomial experiment, there are more than two possible outcomes. However, there are still a fixed number of independent trials, and the probability of each outcome must remain constant from trial to trial.
- Instead of using a combination, as in the case of the binomial probability, the number of ways the outcomes can occur is done using distinguishable permutations.

Example

- The probability that a person will pass a College Algebra class is 0.55, the probability that a person will withdraw before the class is completed is 0.40, and the probability that a person will fail the class is 0.05. Find the probability that in a class of 30 students, exactly 16 pass, 12 withdraw, and 2 fail.

Example

Outcome	**x**	**p(outcome)**
Pass	16	0.55
Withdraw	12	0.4
Fail	2	0.05
Total	**30**	**1**

Example

The probability is found using this formula:

$$P = \frac{30!}{(16!)\,(12!)\,(2!)} * 0.55^{16} * 0.40^{12} * 0.05^{2}$$

Poisson Probabilities

- Named after the French mathematician Simeon Poisson, Poisson probabilities are useful when there are a large number of independent trials with a small probability of success on a single trial and the variables occur over a period of time. It can also be used when a density of items is distributed over a given area or volume.

$$P(x;\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!},\ x = 0, 1, 2, \cdots$$

- Lambda in the formula is the mean number of occurrences. If you're approximating a binomial probability using the Poisson, then lambda is the same as mu or n * p.

Example

- If there are 500 customers per eight-hour day in a check-out lane, what is the probability that there will be exactly 3 in line during any five-minute period?
- The expected value during any one five minute period would be 500 / 96 = 5.2083333. The 96 is because there are 96 five-minute periods in eight hours. So, you expect about 5.2 customers in 5 minutes and want to know the probability of getting exactly 3.
- $p(3;500/96) = e^{(-500/96)} * (500/96)^3 / 3! = 0.1288$ (approx)

Hypergeometric Probabilities

- Hypergeometric experiments occur when the trials are not independent of each other and occur due to sampling without replacement -- as in a five card poker hand.
- Hypergeometric probabilities involve the multiplication of two combinations together and then division by the total number of combinations.

Example:

- How many ways can 3 men and 4 women be selected from a group of 7 men and 10 women?

$$\frac{\binom{7}{3}\binom{10}{4}}{\binom{17}{7}}$$

$= 7350/19448 = 0.3779$ (approx)